MATHEMATICAL ANALYSIS OF DENTAL ARCH OF CHILDREN IN NORMAL OCCLUSION: A LITERATURE REVIEW

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Abstract

AIM. This paper is an attempt to compare and analyze the various mathematical models for defining the dental arch curvature of children in normal occlusion based upon a review of available literature.

Background. While various studies have touched upon ways to cure or prevent dental diseases and upon surgical ways for teeth reconstitution to correct teeth anomalies during childhood, a substantial literature also exists, attempting to mathematically define the dental arch of children in normal occlusion. This paper reviews these dental studies and compares them analytically.

Method. The paper compares the different mathematical approaches, highlights the basic assumptions behind each model, underscores the relevancy and applicability of the same, and also lists applicable mathematical formulae.

Results. Each model has been found applicable to specific research conditions, as a universal mathematical model for describing the human dental arch still eludes satisfactory definition. The models necessarily need to include the features of the dental arch, such as shape, spacing between teeth and symmetry or asymmetry, but they also need substantial improvement.

Conclusions. While the paper shows that the existing models are inadequate in properly defining the human dental arch, it also acknowledges that future research based on modern imaging techniques and computer-aided simulation could well succeed in deriving an all-inclusive definition for the human dental curve till now eluding the experts.

Keywords: Dental Arch, Mathematical Analysis, Normal Occlusion

INTRODUCTION

The primary dentition in children needs to be as close as possible to the ideal, so that, during future adulthood, children may exhibit normal dental features like normal mastication and appearance, dental space and occlusion for proper and healthy functioning of permanent dentition. It is well known that the physical aspect directly impacts on the self-esteem and inter-personal behaviour of the human individual, while dental health challenges – like malocclusions, dental caries, gum disease and tooth loss – do require preventive and curative interventions as early as childhood, so that permanent dentition may be normal in later years. The various parts of the dental arch during childhood, viz., canine, incisor and molar, play a vital role in shaping space and occlusion characteristics during permanent dentition [1]. Prabhakaran et al. [1] even stress the importance of the arch dimensions in properly aligning teeth, stabilizing the form, alleviating arch crowding, and providing a normal overbite and overjet, stable occlusion and a balanced facial profile. Thus, both research aims and clinical diagnosis and treatment have long required the study of dental arch forms, shape, size and other parameters, like over jet and overbite, as also spacing in deciduous dentition. In fact, arch size has been viewed as more important than even teeth size [2]. While various efforts have been made to formulate a mathematical model for the dental arch in humans, its earliest description used terms like elliptic, parabolic, etc. Also, in terms of measurement, the arch circumference, width and depth have been identified by many experts as relevant parameters for measuring the dental arch curve. In fact, some experts have even defined the dental arch curvature through biometry, by measurement of ratios, angles and linear distances [3-10]. Such analysis, however, has serious limitations in describing a three-dimensional (3D) structure like the dental arch [11]. Therefore, numerous mathematical models and geometrical forms have been put forth by various experts, but no two models appear to be clearly defined by means of a single parameter [7]. This paper attempts at highlighting, on the basis of a secondary research on the available literature, the various mathematical approaches to define the curvature of the dental arch of children in normal occlusion. It presents the assumptions, circumstances and limitations of each study, as well as an overall picture of the current stage of research in the
field. It also arrives at the conclusion that further research, particularly by using modern imaging techniques and computer simulation, could well provide a better mathematical model for defining the dental arch of children in normal occlusion.

DEFINING THE DENTAL ARCH

Models for describing the dental arch curvature include conic sections [12,13], parabolas [14], cubic spline curves [15], catenary curves [16], and polynomials of second to eight degree [17], mixed models and the beta function [8]. The definitions differ because of the differences in objectives, dissimilarity of the samples studied and diverse methodologies adopted, which explains why uniform results in defining and arriving at a generalized model factoring in all symmetries and asymmetries of curvature elude experts even today. Some model may be suitable in one case while others may be more so in another situation. In this respect, conic sections, which are 2\textsuperscript{nd} order curves, can only be applied to specific shapes like hyperbolas, ellipse etc., so that their efficiency as an ideal fit to any shape of the dental arch is thus limited [18]. The beta function, although superior, considers only the parameters of molar width and arch depth and does not act in other dental landmarks, nor does it consider the asymmetrical forms. In contrast, the 4\textsuperscript{th} order polynomial functions are more effective in defining the dental arch than either cubic spline or the beta function [18]. AlHarbi \textit{et al.} [18] also maintain that important considerations in defining the human dental arch through mathematical modeling – like symmetry or asymmetry, objective, landmarks used and required level of accuracy – do influence the actual choice of the model.

OCCLUSION AND ITS TYPES

Occlusion is the manner in which the lower and upper teeth intercuspate between each other in all mandibular positions or movements. Ash & Ramfjord [19] state that it is a result of neuro-muscular control of the components of the mastication systems viz., teeth, maxilla & mandibular, periodontal structures, temporomandibular joints and their related muscles and ligaments. Ross [20] also differentiated between physiological and pathological occlusion, in which the various components function smoothly and without any pain, and also remain in good health. Furthermore, occlusion is a phenomenon generally classified by experts into three types, namely: normal occlusion, ideal occlusion and malocclusion.

IDEAL OCCLUSION

Ideal occlusion is a hypothetical state, an ideal situation. McDonald & Ireland [21] defined ideal occlusions as a condition in which maxilla and mandible have their skeletal bases of correct size relative to one another, and the teeth are in correct relationship in the three spatial planes at rest. Houston \textit{et al.} [22] have also given various other concepts relating to ideal occlusion in permanent dentition, referring to ideal mesiodistal & buccolingual inclinations, correct approximal relationships of teeth, exact overlapping of upper and lower arch both laterally and in anterior position, the existence of mandible in position of centric relation, and also the presence of correct functional relationship during mandibular excursions.

NORMAL OCCLUSION AND ITS CHARACTERISTICS

Normal occlusion was first clearly defined by Angle [23], as occurring when upper and lower molars were in a relationship in which the mesiobuccal cusp of the upper molar occluded in buccal cavity the lower molar, and the teeth were all arranged in a smoothly curving line. Houston \textit{et al.} [22] defined normal occlusion as an occlusion within accepted definition of the ideal, causing no functional or aesthetic problems. Andrews [24] had previously mentioned six distinct characteristics observed consistently in orthodontic patients having normal occlusion, viz., molar relationship, correct crown angulation & inclination, absence of undesirable teeth
rotations, tightness of proximal points and flat occlusal plane (the curve of Spee having no more than a slight arch, the deepest curve being of 1.5 mm). To this, Roth [25] added some more characteristics as features of normal occlusion, viz., coincidence of centric occlusion and relationship, exclusion of posterior teeth during protrusion, inclusion of canine teeth solely during lateral excursions of the mandible and prevalence of even bilateral contacts in buccal segments during centric excision of teeth. On the basis of a more recent research study, Oltramari et al. [26] maintain that success of the orthodontic treatments can be achieved when all static & functional objectives of occlusion exist, while achieving a stable centric relation with all teeth in maxim intercuspal position is the main criterion for a functional occlusion.

MATheMATICAL MODELS FOR MEASURING THE DENTAL ARCH CURVE

For detecting future orthodontic problems or for ensuring normal occlusion, a study of the dental arch characteristics becomes essential. Additionally, intra-arch spacing also needs to be studied so that to help dentist forecast and prevent ectopic or premature teeth eruption. While past studies on dentition in children and young adults have shown significant variations among diverse populations [1], dentists are continuously seized of the need to generalize their research findings and arrive at a uniform mathematical model for defining the human dental arch and for assessing the generalizations, if any, in dental shape, size, spacing and other characteristics. Prabhakaran et al. [1] also maintain that such mathematical modelling and analysis during primary dentition is very important in assessing the arch dimensions and spacing, and also for ensuring a proper alignment in permanent dentition during the crucial period which follows the complete eruption of primary dentition in children. They are also of the view that proper prediction of arch variations and state of occlusion during this period can be crucial for establishing ideal desired esthetic and functional occlusion in later years.

While all dentists and orthodontists seem to be more or less unanimous in perceiving the importance of mathematical analysis of the dental arch of children in normal occlusion, no two experts seem to agree on defining the human dental arch by means of a single generalized model. A single model eludes the foremost orthodontic practitioners owing to the differences observed in the samples studied with regard to their origins, size, features, ages etc. Thus, while one author may have studied and derived his results from studying a sample of Brazilian children under some previously defined test conditions, another author may have studied Afro-American children of another age group, sample size or geographical origins. Also, within the same set of samples studied, there are also marked variations in dental arch shapes, sizes and spacing – as found out by leading experts in the field. Shapes are also unpredictable as to the symmetry or asymmetry, and this is another obstacle to the theoretical generalization that could evolve from a single uniform mathematical model. However, some notable studies in the past decades do stand out and may be singled out as the most relevant and significant developments in the field till date.

The earliest models were necessarily qualitative, rather than quantitative. Dentists talked of ellipse, parabola, conic section etc., when describing the human dental arch. Earlier authors, like Hayashi [27] and Lu [28], did attempt to explain mathematically the human dental arch in terms of polynomial equations of different orders. However, their theory could not explain the asymmetrical features or predict fully all forms of the arch. Later on, authors like Pepe [1975] [17], Biggerstaff [1972] [12], Jones & Richmond [1989] [14], Hayashi [1976] [27], BeGole [1980] [15] etc., made their valuable contributions to the literature in the dental field through their pioneering studies on teeth of various sample populations of children in general, and mathematical analysis of the dental arch in particular. While authors like Pepe and Biggerstaff relied on symmetrical features of dental curvature, BeGole was a pioneer in the field, as he utilized the asymmetrical cubic splines to describe the dental arch. His model assumed that the arch
could not be symmetrical and he tried to evolve a mathematical best fit for defining and assessing the arch curve by using cubic splines. BeGole developed a computer FORTRAN program used for interpolating different cubic splines for each subject studied and essentially tried to substantiate a radical view of many experts that the arch curve defied geometrical definition and that perfect geometrical shapes, such as the parabola or the ellipse, could not satisfactorily define the same. He was of the view that the cubic spline appropriately represented the general maxillary arch form of persons in normal occlusion. His work directly contrasted the efforts of Biggerstaff [12] who defined the dental arch form through a set of quadratic equations, and also of Pepe, [17] who used polynomial equations of less than eight degree to fit on the dental arch curve. In Pepe’s view, there could be supposed to exist, at least in theory, a unique polynomial equation having degree \((n+1)\) or less \((n\) was number of data points) that would ensure exact data fit of points on the dental arch curve. An example would be the polynomial equation based on Lagrange’s interpolation formula viz.,
\[
y = \sum_{i=1}^{n} y_i \frac{(x-x_i)}{(x-x_j)},
\]
where \(x_i, y_i\) were data points.

In 1989, Jones & Richmond [14] used the parabolic curve to explain the form of the dental arch quite effectively. Their effort did contribute to both pre and post treatment benefits based on research on the dental arch. However, Battagel [1996] [16] used the catenary curves as a fit for arch curvature and published the findings in the popular British Journal of Orthodontics, proving that the British researchers were not far behind their American counterparts. Then, Harris [1997] [4] made a longitudinal study on the arch form, while the next year [1998], Braun et al. [8] put forth their famous beta function model for defining the dental arch. Braun expressed the beta function by means of a mathematical equation of the form:

\[
y = 3.0314D \left( \frac{X}{W} + \frac{1}{2} \right)^{0.6} \left( \frac{1}{2} - \frac{X}{W} \right)^{0.6}
\]

According to Braun equation, \(W\) was molar width (in mm), denoting the measured distance between right and left 2nd molar distobuccal cusp points, and \(D\) was the depth of the arch. Quite notably, the beta function was a symmetrical function which did not explain the observed variations in the form and shape in actual human samples studied by others. Although it was observed by Pepe [1975] [17] that 4th order polynomials were actually a better fit than the splines, in later analyses developed in the 1990s, it appeared that these were even better than the beta [18]. In the latter part of the 1990s, Ferrario et al. [6] expressed the dental curve as a 3-D structure. These experts conducted some diverse studies on the dental arch for establishing the 3-D inclinations of the dental axes, assessing arch curves of both adolescents and adults and statistically analysing Monson’s sphere in healthy human permanent dentition. Other key authors, like Burris et al. [9], who studied the maxillary arch sizes and shapes in American whites and blacks, Poggio et al. [11], who pointed out the deficiencies in using biometrical methods in describing the dental arch curvature, and Noroozi et al. [10], who showed that the beta function was solely insufficient to describe an expanded square dental arch form, perhaps, constitute some of the most relevant mathematical analyses of recent years.

Recently, one of the most relevant analyses seems to have been carried out by AlHarbi et al. [18], who essentially studied the dental arch curvature of individuals in normal occlusion. They studied 40 sets of plaster dental – both upper and lower casts – of male and female subjects with ages from 18 to 25 years. Although their samples were from adults, they considered four most relevant functions, namely, the beta function, the polynomial functions, the natural cubic splines, and the Hermite cubic splines. They found out that, whereas polynomials of the 4th order best fit the dental arch exhibiting symmetrical form, the Hermite cubic splines best described the irregular in shape dental arch curves, and were particularly useful in tracking treatment variations. They expressed the opinion at the end of their study of subjects – all sourced,
incidentally, from nationals of Saudi Arabia – that the 4th order polynomials could be effectively used to define a smooth dental arch curve to be further be applied to fabricating custom arch wires or a fixed orthodontic apparatus, which could substantially aid in dental arch reconstruction or even in enhancing aesthetic beauty in patients.

COMPARISON OF DIFFERENT MODELS FOR ANALYSING THE DENTAL ARCH

The dental arch has emerged as an important part of modern dentistry for a variety of reasons. The need for an early detection and prevention of malocclusion is an important reason, as dentists hope to ensure a normal and ideal permanent dentition. Dentists also increasingly wish to facilitate normal facial appearance in case of teeth and space abnormalities in children and adults. What constitutes the ideal occlusion, ideal intra-arch and adjacent space and correct arch curvature is a matter of comparison among leading dentists and orthodontists.

Previous studies analyzing dental arch shape have used conventional anatomical points on incisal edges and on molar cusp tips for classifying the forms of the dental arch through various mathematical forms: ellipse, parabola, cubical spline, etc., as already mentioned in the foregoing paragraphs. Other geometric shapes used to describe and measure the dental arch include the catenary curves. Hayashi [27] used mathematical equations of the form: $y = ax^n + e^{al(x-\beta)}$ and applied them to anatomic landmarks on buccal cusps and incisal edges of numerous dental casts. However, the method was complex and required estimation of parameters like $a$, $\beta$ etc. Also, Hayashi did not consider the asymmetrical curvature of the arch. In contrast, Lu [28] introduced the concept of fourth degree polynomial for defining the dental arch curve. Later on, Biggerstaff [12] introduced a generalized quadratic equation for studying the close fit of shapes like the parabola, hyperbola and ellipse, for describing the form of the human dental arch. However, sixth degree polynomials ensured a better curve fit, as mentioned by Pepe [17]. Many authors – like Biggerstaff [12] – have used a parabola of the form $x^2 = -2py$ for describing the shape of the dental arch, while others – like Pepe [17] – have stressed on the catenary curve form defined by equation $y = (e^x + e^{-x})/2$. Biggerstaff [12] has also mentioned equation $(x^2/b^2) + (y^2/a^2) = 1$, that defines an ellipse. BeGole [15] even developed a computer program in FORTRAN, used to interpolate a cubic spline for individual subjects who were studied to effectively find out the perfect mathematical model defining the human dental arch. The method of BeGole essentially utilized the cubic equations, and the splines used in analysis were either symmetrical or asymmetrical. Another method, finite element analysis, used in comparing dental-arch forms, was affected by the homology function and the drawbacks of element design. Multivariate principal component analyses, performed by Buschang et al. [29] to determine the size and shape factors from numerous linear measurements, could not satisfactorily explain the major variations in dental arch forms, so that the method failed to provide a larger generalization for explaining them.

ANALYSING DENTAL ARCH CURVE IN CHILDREN IN NORMAL OCCLUSION

Various studies have been conducted by different experts for defining human dental arch curves by a mathematical model whose curvature has assumed importance, particularly in the prediction, correction and alignment of dental arch in children with normal occlusion. The study of children in primary dentition led to some notable advances in dental care and treatment of various dental diseases and conditions, although an exact mathematical model for the dental arch curve is yet to be arrived at. Some characteristic features that have emerged during the course of various studies over time indicate that no single arch form could be found to relate to all types of samples studied, since the basic objectives, origin and heredity of the children under study, the drawbacks of the various mathematical tools etc., do inhibit a satisfactory and
perfect fit of any model in describing the dental arch form to any degree of correction. However, it has been evident through the years of continuous study developed by dentists and clinical orthodontists that children exhibit certain common features during their childhood, when their dentition is yet to develop into permanent dental form. For example, a common feature is the eruption of primary dentition in children, generally following a fixed pattern. The eruption time of various teeth, like incisors, molars, canines etc., follows this definite pattern over the growing up years of the child. The differences in teeth form, shape, size, arch spacing and curvature etc., that characterize a given sample under study for mathematical analysis, also essentially vary with the nationality and ethnic origin of a child. In a longitudinal study performed by Hendrikson et al. [30] on 30 children of Scandinavian origin with normal occlusion, it was found out that when children pass from adolescence to adulthood, a significant lack of stability in arch form was discernible. In another study, experts have also indicated that dental arches in some children were symmetrical, while in others – not, indicating that a symmetrical form of a dental arch was not a prerequisite for normal occlusion. All these studies, based on mathematical analysis of one kind or another, have thrown up substantial data, yet without achieving a high degree of correlation so as to deliver a generalized theory that can satisfactorily associate a single mathematical model for all dental arch forms in children with normal occlusion.

The various research studies point to different mathematical models as a better fit for defining the human dental arch, which is perhaps due to the differences in perspectives, research conditions, basic assumptions and varied sample types and sizes used by various researchers. Thus, while fitting the human dental arch with a mathematical curve, Biggerstaff [12] chose the quadratic equation as appropriate in describing the common dental arch forms, viz. parabola, hyperbola and ellipse. However, Lu [28] chose the 4th degree polynomial as adequately representing such dental arch curves, while another researcher, Pepe [17], found the catenary curves as an inferior fit even for quadratic equations, but also found the 6th order polynomials as a far better fit than the 4th order polynomials. Nevertheless, she did observe some anomalies in her results and felt that the cubic splines could be considered in further research for better defining the human dental arch. Obviously, later research on cubic splines of BeGole [15] was influenced by this earlier research of Pepe [17]. The research mentioned was thus neither conclusive nor able to perfectly define the shape and size of any and every dental arch.

In her study of cubic splines, BeGole [15], on the basis of previous research on arch curve definition, stated that, ideally, the arch form could be represented by a curve that had immense flexibility; this would then enable better fit on any dental arch, which, throughout childhood and youth, was changing and hence could be also altered by dental intervention methods. She felt that the curve must be able to fit all sizes and shapes of the dental arch, on also including asymmetries unlike the usual geometric forms. According to her, the spline was quite useful because of its symmetry, but also because of its versatility in representing all shapes and sizes of dental arch simply by a suitable selection of the knot points which provided the spline its better fit both with regards to asymmetry and to its ability to accommodate diverse arch shapes and sizes. The spline curve, she felt, was actually continuous everywhere and made up of so many data points so that to fit exactly at the subsets of points called knots, where smoothing occurs ultimately, resulting in a mathematically smooth curve. According to BeGole, the cubic spline, when chosen as a mathematical dental form, also effectively eliminates the limitation of straight lines when fit to the posterior segments of the arch, hence offering various advantages for defining the human dental arch, as compared with other methods. Therefore, BeGole, Pepe, Biggerstaff, Lu and others have substantially contributed to the research on the dental arch and to its defining, however, up to now, no study has been able to conclusively establish the shape of the human dental arch as a universal fit for all types of subjects.
CONCLUSION

The factors determining a satisfactory diagnosis in orthodontic treatment include teeth spacing and size, dental arch form and size. Commonly used plaster model analysis is cumbersome, whereas many scanning tools, like laser, destructive and computer tomography scans, structured light, magnetic resonance imaging, and ultrasound techniques are now in use for accurate 3-D reconstruction of human anatomy. The plaster orthodontic methods can be successfully replaced by 3-D models using computer images for obtaining better and more accurate results. Teeth measurements using computer imaging are accurate, efficient and easy to do, and would prove to be very useful in measuring tooth and dental arch sizes, as well as the phenomenon of dental crowding. Mathematical analysis, though now quite old, can be applied satisfactorily in various issues related to dentistry, while the advances in computer imaging, digitalization and computer analysis through state-of-the-art software programs do herald a new age in mathematical modelling of the human dental arch, and could bring in substantial advancement in the field of Orthodontics and Pedodontics. This could in turn usher in an ideal dental care and treatment environment so necessary for countering the lack of dental awareness and prevalence of dental diseases and inconsistencies in children across the world.

References